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POSING MATHEMATICAL PROBLEMS: AN EXPLORATORY STUDY

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In this study, 53 middle school teachers and 28 prospective secondary school teachers worked either individually or in pairs to pose mathematical problems associated with a reasonably complex task setting, before and during or after attempting to solve a problem within that task setting. Written responses were examined to determine the kinds of problems posed in this task setting, to make inferences about cognitive processes used to generate the problems, and to examine differences between problems posed prior to solving the problem and those posed during or after solving. Although some responses were ill-posed or poorly stated problems, subjects generated a large number of reasonable problems during both problem-posing phases, thereby suggesting that these teachers and prospective teachers had some personal capacity for mathematical problem posing. Subjects posed problems using both affirming and negating processes; that is, not only by generating goal statements while keeping problem constraints fixed but also by manipulating the task's implicit assumptions and initial conditions. A sizable portion of the posed problems were produced in clusters of related problems, thereby suggesting systematic problem generation. Subjects posed more problems before problem solving than during or after problem solving, and they tended to shift the focus of their posing between posing phases based at least in part on the intervening problem-solving experience. Moreover, the posed problems were not always ones that subjects could solve, nor were they always problems with "nice" mathematical solutions.

Problem posing is of central importance in the discipline of mathematics and in the nature of mathematical thinking. Some distinguished leaders in mathematics and mathematics education (e.g., Freudenthal, 1973; Polyá, 1954) have identified problem posing as an important part of a student's mathematical experience, and documents promoting curricular and pedagogical innovation in mathematics education (National Council of Teachers of Mathematics [NCTM], 1989, 1991) have recently called for an increased emphasis on problem-posing activities in the mathematics classroom. For example, the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) advocates that students be given increased opportunities for "investigating and

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formulating questions from problem situations" (p. 70), and refers explicitly to problem posing by arguing that "students should also have some experience recognizing and formulating their own problems, an activity which is at the heart of doing mathematics" (p. 138). Included in the *Professional Standards for Teaching Mathematics* (1991) is the idea that "students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem" (p. 95). Despite the importance of problem posing as a form of mathematical activity, and despite interest in its use as an instructional activity, there has been little systematic investigation of mathematical problem posing as a cognitive process involving generating a problem from a situation or an experience.

The term "problem posing" has been used to refer both to the generation of new problems and to the reformulation of given problems (Silver, 1994). One kind of problem posing, usually referred to as problem formulation or reformulation, occurs within the process of solving a complex problem when a solver restates or recreates a given problem in some way to make it more accessible for solution. This is the form of problem posing that prompted Duncker (1945) to comment about 50 years ago that problem solving consists of successive reformulations of an initial problem. Since that time, problem formulation has been extensively studied by researchers interested in understanding complex problem solving, and it has become increasingly common to view problem solving as a process involving establishing a series of successively more refined problem representations that incorporate relationships between the given information and the desired goal, and into which new information is added as subgoals are satisfied. In fact, one of the major findings of an extensive body of research on the differences between experts and novices in a variety of complex task domains is that experts tend to spend considerable time engaging in problem formulation and reformulation, usually engaging in qualitative rather than quantitative analysis, in contrast to novices who spend relatively little time in formulation and reformulation (Silver & Marshall, 1989).

When the term problem posing is used in contemporary mathematics education reform documents (e.g., NCTM, 1989, 1991), however, it usually refers to a somewhat different kind of activity, in which problem posing itself is the focus of attention. In this case, the goal is not the solution of a given problem but the creation of a new problem from a situation or experience. Such problem posing can occur prior to any problem solving, as would be the case if problems were generated from a contrived or naturalistic situation. This type of problem generation is also sometimes referred to as problem formulation, but the process being described here is different from the reformulation that occurs within complex problem solving itself. Problem posing can also occur after solving a particular problem, when one might examine the conditions of the problem to generate alternative related problems. This latter form of problem posing is associated with the "Looking Back" phase of problem solving discussed by Polyá (1957).

Although the forms of activity being advocated in current calls for mathematics instructional reform have been subjected to far less research scrutiny than has the process of problem reformulation within problem solving, some interesting instructional explorations involving problem posing have been undertaken (e.g., Brown & Walter, 1990; Hashimoto, 1987; Healy, 1993; Keil, 1965; Skinner, 1991; van den Brink, 1987;

Winograd, 1991). For example, Brown and Walter (1990) have written extensively about a version of problem posing in which problem conditions and constraints are examined and manipulated through a process they refer to as "What-if-not?" These explorations have suggested some productive approaches toward the integration of problem posing into mathematics classroom instruction but that there has been almost no systematic research conducted on mathematical problem posing as it occurs prior to or after problem solving, and little is known about the nature of problem posing as a cognitive process (Kilpatrick, 1987). Therefore, this exploratory study was undertaken to provide some information about the nature of problem posing as a complex cognitive process.

Two basic questions were explored in this study: What are the kinds of problems posed by people within a reasonably complex task setting? What are the differences, if any, between the kinds of problems people pose in that setting prior to solving a problem embedded in that setting and the kinds of problems posed in the setting during or after solving the problem? In order to illuminate mathematical problem-posing processes, it would have been reasonable to conduct interviews with a few selected subjects and to analyze their verbal protocols. However, when we found that it would be possible to collect data from a large number of subjects, we chose to have subjects respond in writing rather than in interview settings. Furthermore, because others have used written data successfully to uncover cognitive process information about mathematical problem solving (e.g., Hall, Kibler, Wenger, & Truxaw, 1989), we decided that it would be appropriate to use paper-and-pencil data here as the basis for an analysis of mathematical problem posing.

The subjects in this investigation were in-service middle school mathematics teachers and preservice secondary school mathematics teachers. Because current reform documents have suggested the importance of problem posing, and because this type of activity has not been a feature of conventional mathematics instruction, it seemed reasonable to examine the capacity of teachers themselves to engage in the process of problem posing. If actual and prospective teachers exhibit a generative capacity in their own mathematical activity, then it is reasonable to expect that a lack of personal competence will not be a major obstacle to their incorporating problem-posing activities into their teaching.

METHOD

Subjects

The subjects were 53 middle school mathematics teachers and 28 preservice secondary school mathematics teachers. The middle school teachers were participants in a week-long mathematics teaching workshop sponsored by their school district in Summer 1988; their formal mathematics background ranged from having an undergraduate degree in mathematics (one teacher) to having almost no formal college-level mathematics coursework (12 teachers), and their teaching experience ranged from 2 years to more than 20 years. The preservice secondary mathematics teachers were enrolled in a mathematics teaching methods course at a public university in Fall 1989; all had recently completed substantial coursework in mathematics (essentially an undergraduate major). Given the nature of the performance studied here, the differences between the two groups in the extent and recency of their subject-matter knowledge and in their teaching experience were thought to be of offsetting importance in influencing performance. That is, although differences in formal subject-matter knowledge and in the recent study of mathematics favored the preservice secondary teachers, the specific requirements of the task used in this study were closely aligned with the mathematics content commonly taught in middle school rather than that typically studied in college. Thus, results are reported here only for the aggregated sample. A report of preliminary analyses of data obtained from the middle school teacher sample is provided by Silver and Mamona (1989).

The Task and Administration

The Billiard Ball Mathematics (BBM) task consisted of three parts, each of which is shown in Figure 1. The BBM task was presented to all subjects in exactly the same way. In the first and third parts, subjects were asked to pose problems related to a task setting in which a billiard ball is projected from the lower left corner of a rectangular table at an angle of 45° to the sides; in the middle part of the task, they were asked to solve a particular nontrivial problem related to this task environment. Subjects completed all parts of the BBM task in 45 minutes. They were given 10 minutes to generate problems in the first phase, Initial Posing (IP), and 30 minutes to solve the problem in the second phase, Problem Solving (PS). The final phase, Additional Posing (AP), coincided with the 30-minute PS phase, during which time subjects recorded problems generated during problem solving, and an additional 5 minutes after the PS phase, during which time they could generate additional problems related to the task setting.

The BBM task was adapted for use as a problem-posing and problem-solving task from versions that exist in published sources for use as a problem-solving task with middle school students (e.g., the "Paper Pool Activity" in Fitzgerald, Winter, Lappan, & Phillips, 1986) and with secondary school and college students (e.g., the pool table problem in Jacobs, 1970). The task was thought to be an environment rich enough to permit the posing of interesting problems and conjectures, yet one in which the required problem solving would be possible, because it required only knowledge of rather simple mathematical concepts (e.g., factors, multiples, ratios).

Each subject either worked on the BBM task individually or as a member of a pair. Of the 53 middle school teachers, 25 worked individually and 28 worked in 14 pairs. Of the 28 preservice secondary teachers, 8 worked individually and 20 worked in 10 pairs.

RESULTS

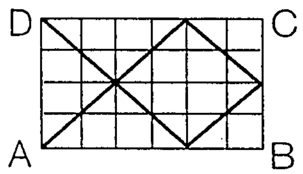
A total of 399 responses were generated in the problem-posing phases (IP and AP) of the BBM task. In examining the nature of a subject's response, the written response was considered along with any accompanying diagrams or drawings made on accompanying pages. Because this report is concerned with mathematical problem posing, only the results for the two problem-posing phases (IP and AP) are presented in detail here. Results for the problem-solving phase (PS) are mentioned only briefly as they relate to interpreting the problem-posing findings.

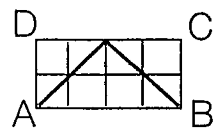
A

(Part 1)

Imagine billiard ball tables like the ones shown below. Suppose a ball is shot at a 45° angle from the lower left corner (A) of the table. When the ball hits a side of the table, it bounces off at a 45° angle.

In each of the examples shown below, the ball hits the sides several times and then eventually lands in a corner pocket. In Example 1, the ball travels on a 6-by-4 table and ends up in pocket D, after 3 hits on the sides. In Example 2, the ball travels on a 4-by-2 table and ends up in pocket B, after 1 hit on the sides.





Look at the examples, think about the situation for tables of other sizes, and write down any questions or problems that occur to you.

B

(Part 2)

[NOTE: The first two paragraphs and the examples from IP phase repeated]

Look at the examples, think about the situation for tables of other sizes, consider as many examples as you need, and try to predict the final destination of the ball. That is, when will the ball land in pocket A? When will it land in pocket B? In pocket C? In pocket D?

C

(Part 3)

As you work out your solution to the problem, other questions may also come to mind. In the space provided below, write down any questions or problems that occur to you.

Figure 1. The billiard ball mathematics (BBM) task.

Initial Examination of Posed Problems

Most of the subjects' responses were expressed as problems or questions regarding the path or destination of the billiard ball on tables of varying sizes, the effect of varying the task's given conditions, or the underlying assumptions of the task. The following are representative of these types of responses: "If the table dimensions were decreased by 2, will the number of hits decrease by 2?" "Would the ball end up in pocket C if the table were square?" "What is the pocket and number of hits for a 6×3 table?" "What would happen if the angle were different, like 60°?" More than 60% of the responses were stated clearly as problems or questions, expressed as complete or nearly complete sentences, with a connection to the BBM task.

Another set of responses (about 25% of the total) were expressed not in the form of questions but rather in the form of conjectures. Responses such as the following are representative of the kinds of conjectures generated by subjects: "The larger the table, perhaps the more bounces off the side." "It seems that the ball will end up in the pocket in $a + b - 2$ ricochets on an $an \times bn$ table, provided a and b are relatively prime." "Square tables take zero hits on the sides." "Table sizes and hits are related proportionally." "Size of table is not a factor." As can be seen from these examples, conjectures were sometimes stated in a tentative form and sometimes as definitive assertions (though not always correct, because "Size of table is not a factor" is clearly a false assertion). Nevertheless, these statements were treated as appropriate responses to the BBM task request to generate questions or problems related to the situation, because each assertion can be taken to represent an implicitly stated problem that either was or could be investigated further.

The remaining responses (about 15%) were judged not to be appropriate responses to the BBM task, and these responses were eliminated from further consideration. Some of these eliminated responses may have represented reasonable thinking on the part of the subject, but the written information was far too ambiguous to allow interpretation. Many of the ambiguous responses were expressed as isolated words or phrases, such as the following examples: "square? isosceles triangle?" "points, start, hit, form a triangle (1 hit)"; and "If you stay with the sequence of units of." The other responses in the set eliminated from further consideration were meta-level comments, such as the following: "I feel frustrated because I am having trouble finding a pattern." "Can protractors be used?" "How can I organize all this info?" "What do they want solved?" "To tell you the truth, I'm confused!" "Not familiar with the game of pool." and "This seems like a progression problem, but I just didn't have enough time to figure it out."

Although there was considerable variety in the nature and form of written responses given by subjects in the IP and AP phases, most were appropriate responses that represented explicitly or implicitly posed problems. Therefore, a more extensive analysis was conducted on the 334 appropriate responses (i.e., all except the meta-level comments and the ambiguous statements).

Commonly Posed Problems

Given the variety of ways in which subjects in this study expressed their posed problems, determining when the responses of two different persons expressed essentially the same underlying problem was a nontrivial task, and there were relatively few instances in which the responses of two different persons matched exactly. Moreover, some responses were vague and somewhat difficult to interpret. Nevertheless, subjects' responses could be grouped into clusters corresponding to prototype problems in order to examine the types of problems posed. In this way we found that a majority of the posed problems dealt with relationships between and among the table dimensions (length and width), the number of times the ball hits the sides on its path to its final destination, and the final pocket the ball enters. Problems such as the following were prototypical:

- What is the relationship between the length and width of the table and the number of times the ball hits the sides?
- What is the relationship between the length and width of the table and the final pocket the ball enters?
- What is the relationship between the number of hits and the final pocket the ball enters?
- What is the number of hits (or final pocket the ball enters) when the table dimensions are 6×5? Both odd numbers? Both even numbers?
- What is the number of hits (or final pocket the ball enters) when the table is square?

Responses associated with this type of problem were thus fairly closely associated with the kinds of relationships between table dimensions and number of hits or final pocket that underlie problems that are typically posed for students when the BBM task setting is used in curriculum materials (e.g., Fitzgerald et al., 1986; Jacobs, 1970). Nevertheless, the responses produced by subjects in this study varied in degree of generality or specificity. Some responses were stated in a very general way (e.g., Given an $M \times N$ table, in which pocket will the ball land?), and others were stated with less generality (e.g., For a 3×5 table, how many hits?).

The remainder of the posed problems tended to deal with other aspects of the BBM task setting (e.g., initial angle, characteristics of table, ball, or path traveled). The following are prototypical examples of these kinds of responses:

- What happens if the angle is different from 45° (e.g., 60°, 110°)?
- How does the speed (or velocity) of the ball affect the outcome?
- What happens when spin ("English") is put on the ball?
- Will the ball always land in a pocket? (or, Will the ball never land in a pocket?)
- Will a ball shot from pocket A ever land in pocket A?
- What happens if the ball is shot from a pocket other than A?
- Excluding dimensions, what are the characteristics of the table? (e.g., How many pockets? Where are the pockets located? Is the table level and flat?)
- Will the angle of incidence always equal the angle of reflection?
- How does the force of the shot affect the outcome? (e.g., What happens if you hit the ball too hard? Too easy?)
- What are the characteristics of the path the ball follows? (e.g., In which direction does the ball travel on an 8-by-6 table?)

Responses associated with this type of problem were not at all like the kinds of tasks that are typically posed for students when the BBM task setting is used in curriculum materials (e. g., Fitzgerald et al., 1986; Jacobs, 1970). Moreover, many of these responses indicated a concern with the practical aspects of pool tables (e.g., location of extra pockets, flatness) or the path of pool balls (e.g., spin, friction), thereby suggesting that some subjects were treating the task as practical rather than abstract.

Differences in Posing Problems in the IP and AP Phases

Table 1 displays the mean number of posed problems for each posing phase (IP and AP) of the BBM task for subjects who worked individually and those who worked in pairs. Subjects posed an average of about four problems in the IP phase; the difference between the average number of problems posed in the IP phase by individuals and pairs was not statistically significant. There were fewer problems posed by individuals and by pairs in the AP phase than in the IP phase, but the difference was statistically significant only for the responses of pairs ($t(46) = 3.6, p < .01$). Within the AP phase, individuals posed an average of about one more problem than pairs, but this difference was not statistically significant.

Table 1
Mean Number of Posed Problems by Posing Phase for Individuals and Pairs

	IP phase	AP phase
Individuals ($n = 33$)	3.7 (1.9)	2.6 (2.9)
Pairs ($n = 24$)	3.7 (1.8)	1.6 (2.2)

Note. Standard deviations are in parentheses.

Beyond a general examination of response frequency in the two task phases, it is also interesting to consider the frequency with which particular problems were posed in each phase. A careful consideration of particular responses in each task phase has the potential to reveal important aspects of subjects' thinking while posing problems. There were some interesting similarities and differences between the IP and AP phases both with respect to the generality of the posed problems and to the frequency with which certain types of problems were posed, and these differences appear to be due to the influence of the intervening problem-solving phase. For example, in both the IP and AP phases, subjects frequently posed problems involving table dimensions, the number of hits, and the final pocket. More such problems were posed in very general form in the IP phase, but these generally stated problems represented about the same proportion of problems posed in both the IP and AP phases.

The lack of an increase in stating problems in general form is somewhat surprising, because it is reasonable to assume that the intervening problem-solving phase, in which a fairly general problem was the target of solution, would have encouraged generalization. Nevertheless, the nature of the problem solving done by most subjects probably mitigated against a trend toward greater generality in the AP phase. In particular, most subjects were not able to solve the problem completely in the allotted time for the PS phase, and most attempted a solution through a process of examination of specific cases in an effort to find patterns and generate generalizations. Thus, it is likely that the combination of attempting to solve by examining specific cases and failing to achieve a general solution actually influenced subjects to pose problems with less generality than expected. This explanation is fortified by the finding that the solution attempts produced by subjects generally contained a solution or partial solution for pockets B, C, and D but not for pocket A, and problems concerned with getting the ball (which was originally shot from pocket A) to return to pocket A became much more prevalent in the AP phase than in the IP phase. Only 6 problems dealing with pocket A were posed in the IP phase, but 24 such problems were posed in the AP phase. In fact, the problem concerned with getting the ball into pocket A was the most frequently posed (and often the only) problem in the AP phase for subjects working in pairs.

The Process of Posing Problems

In posing problems during the IP and AP phases, some responses indicated that subjects generated problems by keeping the problem constraints fixed and focusing their attention simply on generating goals. Such problems involved either a specific goal (e.g., determining the number of hits or the final destination of the ball for a table of specified dimensions) or a general goal (e.g., seeking a relationship between the size of the table and the number of hits or the final destination of the ball). In this process of generating problems, one "accepts the given" (Brown & Walter, 1990, p. 15), but other responses also suggested that another process was used to generate problems. Some responses indicated that subjects manipulated the given constraints of the task setting as they generated goals, using a process Brown and Walter call "challenging the given" (1990, p. 15). These problems involved either changing the underlying assumptions in the BBM task (e.g., introducing spin on the ball, introducing or removing friction as a consideration, varying the ball's speed or momentum, questioning the relationship between angle of incidence and the angle of reflection) or on changing the explicitly stated conditions of the task (e.g., shooting the ball at an angle other than 45°, moving the starting point from the lower left corner to another position, changing the number of pockets). Each of the 334 posed problems could thus be categorized as a Goal problem (GL) or as one of two types of constraint-manipulation problems: Initial Conditions (IC) or Implicit Assumptions (IA).

The posed problems were coded by two raters working independently. For the purpose of establishing interrater reliability, about 75% of the posed problems were coded by both raters. Their interrater agreement was quite high (Kappa > .90), and the few disagreements were resolved through discussion to reach consensus.

Table 2 shows the frequency of problems posed in each category for individuals and pairs in each posing phase. Overall, about 60–70% of the problems posed were classified as goal problems; within the 30–40% of the problems classified as constraint-manipulation problems, they were about equally divided between those involving changes in the underlying assumptions and the initial conditions of the task setting. A similar proportional distribution of posed problems into these categories was observed in both posing phases. There was no significant difference between the proportions of goal and constraint-manipulation problems posed by subjects working in pairs or individually, although the pattern across the two kinds of constraint-manipulation problems was significantly different for pairs and individuals in the two phases ($\chi^2[2, N = 125] = 6.36; p < .05$).

Table 2
Frequency of Posed Problems in Each Category for Each Posing Phase for Individuals and Pairs

	Constraint manipulation		Goal generation	
	Implicit assumptions (IA)	Initial conditions (IC)	Goals (GL)	Total
IP phase				
Individuals	23	23	75	121
Pairs	14	22	50	86
AP phase				
Individuals	9	19	60	88
Pairs	11	3	25	39
Total	57	67	210	334

Although problems generated via constraint manipulation were fairly common in the set of responses, very few subjects generated their problems solely from this perspective. If each set of IP responses and each set of AP responses for the 33 individuals and 24 pairs is considered as a unit, only about 9% (10 of the 114 sets of responses) contained only problems generated via constraint manipulation. In contrast, about 33% (38 of the 114 sets of responses) contained only problems generated by keeping the constraints fixed and posing new goals. The proportions of "pure" constraint manipulation responses and "pure" goal generation responses were quite similar in the IP and AP phases.

Relationships Among Posed Problems

The posed problems were also examined for evidence of possible relationships among clusters of problems posed by each subject in order to detect underlying cognitive processes. Several different kinds of relatedness were detected in the problems, the most prominent of which were *chaining* and *systematic variation.*

One kind of relatedness evident in subjects' responses was called chaining, because the set of related problems appeared to have a sequentially linked character. A form of chaining occurred when the answer to one problem was needed in order to generate the answer to the next problem in sequence. The most commonly observed form of chaining, illustrated in the set of problems denoted Cluster A in Figure 2, involved a cluster in which the first few problems were structured so as to lead to (or be closely associated with) a generalization stated or implied in the first or last problem posed in the chain. A somewhat different kind of chaining relationship is illustrated in the set of problems designated as Cluster B in Figure 2, in which the first three problems (stated in the form of conjectures) undergird the final problem (also stated in the form of a conjecture).

Cluster A

Where does the ball land for a 2×4 table?

Where does it land for a 3×6 table?

Where does it land for a 4×8 table?

Where does it land for a table with length twice as long as width?

Cluster B

Assuming no spin, a table that is $2n \times n$ will always end up with the ball in the B pocket with one ricochet.

A table $3n \times 2n$ will always end up with the ball in pocket D with three ricochets.

A table $4n \times 3n$ will always end up with the ball in pocket B after five ricochets.

It seems that the ball will end up in the pocket in $a + b - 2$ ricochets on an $an \times bn$ table, provided a and b are relatively prime.

Cluster C

What happens [How many hits] in the case of a square table?

What happens when $l = 2w$?

What happens when $l = 3w$?

Cluster D

What if the table were square?

What if the angle were not 45 degrees?

What if the ball had not originally been shot from a corner?

What if the ball is shot with a different initial force?

What if the ball spins?

Figure 2. Examples of clusters of related problems.

Other problem clusters contained problems that were considered to be related in another way, as systematic variants of each other. In this type of relatedness, a critical aspect of a problem is held constant while other critical aspects are varied systematically. The following pair of posed problems is an example of relatedness by systematic variation:

> What is the relation of the table dimensions to the final pocket?
> What is the relation of the table dimensions to the number of hits?

Here, concerns about table dimensions remain constant across the pair, and the critical features of final pocket and number of hits are varied between the pair. Systematic clusters were occasionally larger than pairs of problems, as is evident in the set of posed problems designated as Cluster C in Figure 2. In this example, systematic variation is evident in a constant concern about an outcome (it is fairly clear from the other problems posed by this subject that what was meant here by "what happens" was "how many hits occur") when the dimensions of the table are varied.

Other examples of relatedness were also found in subjects' responses. For example, a few pairs of problems illustrated a type of relatedness that might be called *symmetry*, in which the goals and conditions of one problem are symmetrically exchanged in the other problem:

> Given the number of hits and the final pocket, can you determine the dimensions of the table?
> Given the dimensions of the table, can you determine the number of hits and the final pocket?

Another set of related problems, designated Cluster D in Figure 2, was posed by a pair of subjects working together, and it is evocative of Brown and Walter's what-if-not problem-posing process (1990). The relatedness here is based on a common tendency to challenge implicit or exlicit constraints and to state the problems in very open-ended form. A somewhat more general version of this kind of problem relatedness was evident when all (or nearly all) the problems generated by an individual or pair appeared to be focused on a singular set of concerns, such as generating problems dealing exclusively with the set of implicit assumptions in the task or with the feasibility of having the ball return to the pocket from which it was originally shot.

More than half of the subjects generated problems that gave explicit evidence of at least one type of problem relatedness. The responses of individuals were about 50% more likely than those of pairs to show evidence of relatedness among clusters of posed problems.

DISCUSSION

In this study, middle school teachers and prospective secondary school teachers worked individually or in pairs to pose mathematical problems associated with a reasonably complex task setting, before and during or after attempting to solve a problem within that task setting. In response to the BBM problem-posing task used in this study, subjects were able to generate a large number of reasonable problems during both problem-posing phases, thereby suggesting that these teachers and prospective teachers have some personal capacity for mathematical problem posing. In fact, almost all subjects successfully posed at least one problem in both posing phases. These findings suggest that middle school and secondary school teachers are able to engage in reasonable ways with mathematical problem posing, thereby suggesting that their own lack of substantial educational experience with problem posing should not be a barrier to their being able to use problem posing with their students. Nevertheless, many responses were also ill-posed or poorly stated problems. For example, a sizable portion of the 15% of BBM task responses that were excluded from analysis were too ambiguous to allow adequate interpretation, and even among those judged to be adequate, it was sometimes necessary to be quite generous in interpreting the meaning of the responses.

Given that the subjects in this study were either middle school mathematics teachers, presumably accustomed to developing or providing problems for their students, or preservice secondary mathematics teachers with experience in doing university-level mathematics, the frequency of inadequately stated problems is quite disappointing. Thus, there appears to be a need to provide more opportunities for prospective and in-service teachers to engage in mathematical problem posing and to analyze the emerging problems for their feasibility and their quality. As teachers become more proficient in their own problem posing, it is reasonable to assume that they will become more willing to have their students engage in such activities.

It was hoped that this study would suggest some interesting ways in which collaboration might influence mathematical problem posing. In fact, very few differences were detected between the responses to the BBM task by individuals and pairs, and the only difference of note was that subjects working individually gave more evidence of relatedness among clusters of posed problems than subjects working in pairs. The general findings, and the particular result regarding relatedness, suggest that the pairs may not have been functioning well as collaborators in their problem posing, perhaps because the two persons in each pair were not accustomed to working together or perhaps due to the novelty of the problem-posing task. If a pair functioned as two individuals working in parallel, then one would expect to find little evidence of relatedness within the set of posed problems, and this is what was found in this study. Thus, the possible influence of collaboration on mathematical problem posing awaits further investigation.

Although generalization from the results of this study is limited by the fact that the results are based on written responses to a single problem-posing task, there are nevertheless several findings that appear to illuminate aspects of problem posing as a cognitive process and suggest the feasibility and value of further research in this area. For example, it was found that subjects spontaneously engaged not only in "accepting the givens," when they posed problems by keeping the implicitly and explicitly given constraints fixed and simply generating goals, but also "challenging the givens," when they varied the initial conditions or implicit assumptions of the given task setting. Almost 40% of the total number of posed problems gave evidence of subjects' readiness to manipulate the implicit assumptions or explicit conditions of the task, yet only 9% of the sets of responses generated by subjects in both posing phases of the BBM task contained problems posed exclusively in this way. Thus, the data from this study suggest that many subjects engaged in such behavior at least once in their problem posing, and the large number of problems apparently posed in this way suggests the likelihood that most of the subjects in this study might be receptive to the what-if-not instructional approach to problem posing (Brown & Walter, 1990), in which the conditions and constraints of a problem are systematically varied as a means of generating new problems, because they spontaneously engaged in such an approach in their own problem posing.

The extent to which people are willing to engage in challenging the givens, and the relationship between this tendency and receptivity to what-if-not instruction may both be fruitful areas for further research. If such research were undertaken, however, it would be wise also to consider another finding of this study; namely, that subjects tended to generate more problems in the Initial Posing (IP) phase than in the Additional Posing (AP) phase. This result may suggest a natural tendency or preference for problem generation prior to problem solving, thereby suggesting a possible complication in implementing the what-if-not instructional approach, which relies on post-hoc problem generation. It is possible, however, that the differential problem-posing frequency in the two phases may be due as much to task and time constraints as to natural tendencies in human problem posers, because most subjects did not generate any problems during the problem-solving portion of the time available for the AP phase, thereby leaving only 5 minutes for problem posing for the AP phase (i.e., half the time available for the IP phase). Nevertheless, this finding should be considered and subjected to further examination in future research related to post-hoc problem posing.

Another potentially interesting process-related finding is the suggestion that subjects' problem posing may have been influenced not only by their mathematical knowledge but also by other experiences in related task settings. For example, some subjects posed problems such as "Will the angle of incidence always equal the angle of reflection?" and "What effect does friction have on the outcome?" Was their posing affected by a perception of the BBM task as an applied physics problem? Other subjects posed problems such as "What happens if you move the shooter to another pocket?" and "What happens if you put 'English' (spin) on the ball?" Was their posing affected by their recreational experiences in playing pool? Although the written response data alone do not allow us to know with certainty if subjects were actually trying to apply physical principles related to the playing of pool or to the movement of objects on frictionless surfaces, or if the task evoked in them experiences and ideas related to what they may have perceived as impediments or difficulties (literally, *problems*) that might interfere with the idealized path of the ball in the BBM task. Nevertheless, the influence of such prior experience seemed evident in the problem posing of some subjects. Understanding the effects of prior experiences on mathematical problem posing appears to be a promising area for further investigation.

Yet another important process-related consideration in problem generation is the extent to which subjects gave evidence of being systematic in posing problems. As Kilpatrick (1987) has noted, there are many cognitive mechanisms, such as reasoning by analogy, that could be used to generate new problems. In this study, it was possible that subjects could engage in problem posing by generating problems through an essentially random process of goal generation or constraint manipulation, or it was possible for subjects to generate clusters of related problems. The finding that more than half of the subjects generated problems that gave evidence of a least one type of problem relatedness (systematic variation, chaining, symmetry) strongly suggests that much, although certainly not all, of the problem posing by novice problem posers in this study was done in a systematic manner. If this tendency to be fairly systematic can be found in other populations and across many tasks, this result may have important instructional implications, because systematic approaches to problem posing, such as Brown and Walter's what-if-not instruction, could be seen as related to the informal approaches taken by novice problem posers.

Instructional relevance can also be found in the results regarding the correspondence between the problems generated by subjects in this study and the problems that are typically included when the BBM task setting is included in curriculum materials for middle school and high school students (e. g., Fitzgerald et al., 1986; Jacobs, 1970). Some problems routinely included in curriculum materials related to the BBM task setting were also ones that were posed by the subjects in this study. For example, problems regarding the relationship between table size and the final destination of the ball or the number of hits were not only quite commonly posed by subjects, but they also appear in the curriculum materials. On the other hand, other problems found in the curriculum materials were rarely or never generated by subjects in this investigation. For example, the problem of determining a relationship between table size and the distance traveled by the ball during its path around the table is found in the curriculum materials, but *not one* of the 33 individuals or 24 pairs in this study generated this problem.

This set of findings appears to be important for two related reasons. From a pragmatic instructional perspective, it seems important to know that some problems typically included in curriculum materials might be fairly "natural" for subjects to pose for themselves, if they were given an opportunity to do so (e.g., in this case, determining the relationship between table size and final destination of the ball). Because students may be more highly motivated to solve a problem if they have posed it for themselves rather than having it posed by an external source, there may be instructional advantage to providing students with opportunities to pose problems for themselves, whenever it is feasible to do so. As was done here with the BBM tasks, many standard problem-solving activities could be similarly transformed into more open-ended explorations involving problem posing and problem solving (Silver, Kilpatrick, & Schlesinger, 1990). Nevertheless, it is also true that other problems may be less natural for students to pose for themselves (e.g., in this case, determining a relationship between table size and the distance traveled by the ball during its path around the table). Thus, even if teachers provide students with opportunities to generate their own problems, certain problems may not arise naturally from the problem-posing activity of students; thus, some problems may need to be introduced in another way.

Another reason for the importance of these findings is more theoretical, because these results (taken together with others in this study) suggest a complex relationship between problem posing and problem solving. It had been hoped that the design of this study would allow some deep insights into this relationship by affording an opportunity to analyze differences in the problem posing of successful and unsuccessful solvers, and by analyzing differences in the problem solving of those who posed the target problem prior to solution and those who did not. Unfortunately, the PS phase target problem was successfully solved in the allotted time by only a few subjects, and the number was not sufficient to support a careful analysis of the differences noted above. Despite this limitation, the findings of the study did illuminate some aspects of the relationship between problem posing and problem solving.

A general concern is whether or not a person will pose only problems that he or she has already solved or is confident of solving. If so, then one's problem posing could be considered an index of one's problem solving (Kilpatrick, 1987). The finding that some subjects posed problems that would be very difficult (probably impossible) for them to solve suggests that they were not always aware of solutions for their posed problems. For example, not only did some subjects pose the problem about determining the relationship between the table's dimensions and the final pocket, which they were then not able to solve completely in the allotted time, but also some subjects posed problems involving changing the measure of the initial angle from 45° to a different angle measure. Problems in this task setting involving angle measures other than 45° are much more difficult to solve than those involving an initial angle of 45°, and this is why such problems do not appear in curriculum materials for middle school and high school students. In general, the problems in those curriculum materials appear there because they have "neat" mathematical solutions that can be obtained fairly directly from the use of commonly taught elementary mathematical ideas. Thus, the finding that subjects posed problems about varying task constraints like the initial angle suggests that they did not simply pose problems they knew they could solve or for which they had already determined a solution. On the other hand, the frequent posing of conjectures, some of which were accompanied by supporting sketches, suggests that problem posing was not always done in a manner independent of problem solving.

Another finding suggests a clear influence of problem-solving activity on the post-hoc posing of subjects in this study. In particular, the most frequently posed problem in the AP phase concerned getting a ball originally shot from pocket A to return to pocket A, and this problem was posed more than three times as often after problem solving than before. The fact that the problem of getting the ball shot from pocket A to return to pocket A remained an unsolved part of the problem for most persons during the Problem Solving (PS) phase of the BBM task suggests a good reason why this problem became more salient in the post-solution posing. The finding that subjects' problems were not stated with more generality in the AP phase than in the IP phase is also suggestive of a way in which subjects' problem posing was influenced by their problem solving, because the solution attempts tended to involve checking specific cases to generate patterns and because few general solutions were obtained. Thus, further investigation into the differences between posing before and after problem solving is likely to be fruitful, as is other exploration of the general relationship between mathematical problem posing and problem solving.

CODA

It is said that the hallmark of a good exploratory study is that it raises many more questions than it answers. By that criterion, this study was a success. We hope that our initial foray into the largely uncharted wilderness of mathematical problem posing will encourage others to make similar journeys. Mathematical problem posing is central to the discipline of mathematics, and it is also viewed as desirable instructional practice. If our understanding of mathematical activity is to increase and our capacity to improve mathematics instruction is to strengthen, then much more research is needed to develop a deeper understanding of this and related forms of generative cognitive activity.

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