Math and Coding II: Term Exam (113–2)

100 minutes, full mark = 30

Use of your mobile etc. & Internet: Strictly forbidden.
Discussion with other attending students: Strictly forbidden.

Administrative Remarks

- Write your name and student ID on the answer sheet. Put your student ID on the desk.
- Allowed on your desk: student ID card (required), pens/pencils, correction tools (eraser etc.), rulers, drinks, one book, and either a notebook or several sheets of papers not larger than A3. **Other items must be stored in your bags.**
- You cannot wear watches nor electronic devices. **You cannot have them even in your pockets**.
- **After 13:10, the following actions are considered cheating. You may immediately lose your credit.**
  - If non-allowed items (pen cases, foods, poaches, etc.) are found on desks.
  - If you have mobile phones, tablets, or PC, if they are not stored in your bags, or if you use them. They must be in your bags even after you submit your answer sheets.
- No breaks in principle. After 14:00, you may leave after submission. In case of health problems or other issues, call the TA or lecturer. Leaving without permission counts as submission.
- *Any form of academic dishonesty, including chats, additions/corrections after the period, and using your phones, will be treated by NSYSU "Academic Regulations."*

Scientific Remarks

- Show your calculations or thought process for **partial mark**!
- Use English, where mistakes are tolerated. Meanwhile, scientific mistakes are not tolerated.
- If you find any errors or issues in the questions, explain them on your answer sheet, make necessary adjustments on the question, and answer accordingly.
- You may use the following notations and values without definition/declaration.

$\lvert A\rvert$	determinant of a matrix A (equivalent to $\det A$).
$\lVert\vec{v}\rVert$ or $\lvert\vec{v}\rvert$	norm of a vector $\vec{v}$.
I_n, $O_{m,n}$	$n \times n$ identity matrix and $m \times n$ zero matrix.
I, O	identity matrix and zero matrix with shape understood.
A^{T}	transpose of A.
$\overline{A}$	complex conjugate of A.
$A^{\dagger}$	Hermitian conjugate of A.
$\mathbb{R}, \mathbb{C}$	the set of all real/complex numbers.
$\mathbb{R}^n, \mathbb{C}^n$	the set of all n-dimensional real/complex vectors.
$M^{m,n}$	the set of all $m \times n$ real matrices.
$M^{m,n}(\mathbb{C})$	the set of all $m \times n$ complex matrices.
$\mathrm{Re}\, z$	the real part x of a (complex) number $z = x + \mathrm{i}y$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.
$\mathrm{Im}\, z$	the imaginary part y of a (complex) number $z = x + \mathrm{i}y$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$\sqrt{2} \approx 1.414$ $\sqrt{3} \approx 1.732$ $\sqrt{5} \approx 2.236$ $\sqrt{7} \approx 2.646$ $\pi \approx 3.142$ $\mathrm{e} \approx 2.718$
$\log_{10} 2 \approx 0.3010$ $\log_{10} 3 \approx 0.4771$ $\log_{10} 7 \approx 0.8451$ $\ln 10 \approx 2.303$

Answer **[Part I]**–**[Part IV]**. If you still have time, answer either **[Part V]** or **[Part VI]**.

[Part I] Numerical Analysis Fundamentals (9 points)

Answer the problems based on the following text.

> In Python, real numbers are expressed by [**A**] data type. It is based on the IEEE-754 `binary64` format: a real number x is expressed by 64-bit information, one bit for the sign, [**B**] bits for the exponent p, and 52 bits for the mantissa m, as $x = (\text{sign}) \times$ [**C**].
>
> Because the mantissa has limited accuracy of 52 bits, **(P)** the relative accuracy of x is about [**X**] decimal digits. Strange behaviors such as "$0.1 + 0.2 \neq 0.3$", called [**D**] errors, are caused due to this inaccuracy. Meanwhile, underflow errors and overflow errors are due to the limitation of the exponent, $-1022 \leq p \leq 1023$. **(Q)** The largest number `binary64` can handle is about [**Y**].

(1) Write appropriate numbers, expressions, or words for [**A**], [**B**], [**C**], and [**D**].

(2) About the underlined text **(P)**,

- **a)** Write a number for [**X**] and give a mathematical justification (reasoning).
- **b)** Explain the meaning of "relative accuracy of x".
  [Hint: It is nice if you can express it using mathematical expressions.]

(3) About the underlined text **(Q)**,

- **a)** Choose the most suitable number for [**Y**] from 10^{30}, 10^{50}, 10^{100}, 10^{300}, 10^{500}, 10^{1000}.
- **b)** Give a mathematical justification for your choice.

[Part II] Numerical Analysis Concepts (8 points)

Here is a list of keywords appearing this semester. Answer the problems about them.

A) Gauss elimination	**E)** Runge–Kutta	**I)** adaptive step size
B) Newton's method	**F)** Gauss–Seidel	**J)** Runge phenomenon
C) secant method	**G)** cubic spline	**K)** Gibbs phenomenon
D) Heun's method	**H)** trapezoidal rule	**L)** LAPACK

(1) Choose keywords that are related to interpolation. [Answer by the label (A, B, ...).]

(2) Choose keywords that are related to linear algebra.

(3) Choose keywords that are related to integration.

(4) Choose keywords that are related to differential equations.

(5) Choose keywords that are categorized as iterative methods.

(6) Explain advantages of the secant method and when we should use it.

[The exam questions continue on the next page.]

[Part III] Linear Algebra Review (5 points)

(1) Calculate the inner product of $\begin{pmatrix} 1+2\mathrm{i} \\ 2+\mathrm{i} \end{pmatrix}$ and $\begin{pmatrix} 4\mathrm{i} \\ -2\mathrm{i} \end{pmatrix}$.

(2) Diagonalize the matrix $A = \begin{pmatrix} 1 & 0 \\ 4 & -2 \end{pmatrix}$.

[Part IV] Fourier Analysis (8 points)

[Hint: $\exp(x)$ is the exponential function e^x. It satisfies $\exp(\mathrm{i}x) = \cos x + \mathrm{i}\sin x$ for a real number x.]

(1) Calculate $\displaystyle\int x \exp\left(\frac{\mathrm{i}n\pi x}{L}\right) \mathrm{d}x$, where n is an integer and $L > 0$.

(2) Consider the Fourier series expansion of the function $f(x) = |x|$ for $-1 \le x \le 1$. It is given by

$$f(x) = |x| = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right] \qquad (-1 \le x \le 1). \tag{1}$$

a) Show that $f(x)$ is even.

b) Calculate a_0, a_1, and b_1.

c) Calculate a_n and b_n for an integer $n \ge 2$.

The following are extra problems. You may answer **only one** part, namely, either **[Part V]** or **[Part VI]**. If your answer sheet has answers to both parts, both are ignored.

[Part V] Extra problem (1) Fourier Transformation (8 points)

Assume we use a different definition from the textbook of the Fourier transform,

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{2\pi i w x}\,\mathrm{d}x.$$

(1) Find its inverse transform.

(2) Find the Fourier transform of $\exp(-x^2)$ and $\delta(x-1)$ under this definition.

[Part VI] Extra problem (2) Basic Complex Analysis (8 points)

(1) Find $\displaystyle\int_0^{3/2} \delta(x^2 - n)\,\mathrm{d}x$, where n is a positive integer.

(2) Find $\displaystyle\int (\cosh 2x - \sinh 2x)\cos nx\,\mathrm{d}x$, where n is a positive integer.

[This is the end of the exam questions.]