Term Exam 112–2 (Jun. 3, 2024)

100 minutes, full mark = 40

Use of your notebooks/memos/books:	You can bring-in a book by Kreyszig (printed version) or a sheet of A4 paper (not both).
Use of your mobile etc. & Internet:	Strictly forbidden.
Discussion with other attending students:	Strictly forbidden.

Administrative Remarks

- Write your name and student ID on the answer sheet. Put your student ID on the desk.
- Allowed on your desk: student ID card (required), pens/pencils, correction tools (eraser etc.), rulers, drinks, and the brought-in A4 sheet or a printed book by Kreyszig. **Other items must be stored in your bags.**
- **After 13:20, the following acts are regarded as cheating. You may immediately lose your credit.**
  - If non-allowed items (pen cases, foods, poaches, etc.) are found on desks.
  - If your mobile phones, tablets, or PC are not stored in your bags, or if you use them. They must be in your bags even after you submit your answer sheets.
- Breaks are not allowed in principle. After 14:00, you may leave after submission. In case of health problems or other issues, call the TA or lecturer.
- *Any form of academic dishonesty, including chats, additions/corrections after the period, and using your phones, will be treated by NSYSU "Academic Regulations."*

Scientific Remarks

- Include your calculations or thinking process for **partial mark**!
- Use English, where mistakes are tolerated. Meanwhile, scientific mistakes are not tolerated.
- If you notice any errors/issues in the questions, explain the error on your answer sheet, suitably adjust the question, and answer the corrected question.
- You may use the following notations and values without definition/declaration.

$\mathbb{Z}, \mathbb{N}^+$	the set of all integers / all positive integers.
$\mathbb{N}^0$	the set of all non-negative integers.
$\mathbb{R}, \mathbb{C}$	the set of all real / complex numbers.
$\mathbb{R}^n, \mathbb{C}^n$	the set of all n-dimensional real / complex vectors.
$M^{m,n}$	the set of all $m \times n$ real matrices.
$M^{m,n}(\mathbb{C})$	the set of all $m \times n$ complex matrices.
$\lvert A\rvert$	determinant of a matrix A (equivalent to $\det A$).
$\operatorname{Tr} A$	trace of a square matrix A (equivalent to $\operatorname{trace} A$).
$\lVert\vec{v}\rVert$ or $\lvert\vec{v}\rvert$	norm of a vector $\vec{v}$.
$A^\dagger = \overline{A^{\mathrm{T}}}$	hermitian conjugate of A.
$I_n, O_{m,n}$	$n \times n$ identity matrix and $m \times n$ zero matrix.
I, O	an identity matrix and a zero matrix, where its shape is understood.
$A \overset{\text{row}}{\sim} B$	A and B are row equivalent.

$$\sqrt{2} \approx 1.414 \quad \sqrt{3} \approx 1.732 \quad \sqrt{5} \approx 2.236 \quad \sqrt{7} \approx 2.646 \quad \pi \approx 3.142 \quad \mathrm{e} \approx 2.718$$

Answer **[Part A]**–**[Part C]**. If you still have time, answer **[Part D]**.

[Part A] Matrix Diagonalization (10 points)

Consider matrix $A = \begin{pmatrix} \mathrm{i} & 1 \\ -1 & \mathrm{i} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

(1) Discuss whether they are normal or not.

(2) Find all the eigenvalues and corresponding eigenvectors of A and B.

(3) There exists a unitary matrix U such that $U^\dagger AU$ becomes diagonal. Write down U and the diagonal matrix $U^\dagger AU$, showing the derivation.

(4) There exists an invertible matrix P such that $P^{-1}BP$ becomes diagonal. Write down P and the diagonal matrix $P^{-1}BP$, showing the derivation.

[Part B] Special Functions (15 points)

(1) Consider the following functions:

$$\Gamma(\nu+1) \coloneqq \int_0^\infty \mathrm{e}^{-t} t^\nu \,\mathrm{d}t \qquad (\nu > -1), \qquad\qquad I_\nu(x) \coloneqq \sum_{m=0}^\infty \frac{x^{2m+\nu}}{2^{2m+\nu} m!\,\Gamma(m+\nu+1)}.$$

Starting from these definitions, prove the following equations.

(a) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(b) $I_{-n}(x) = I_n(x)$ for positive integers n.

(c) $I_{1/2}(x) = \sqrt{\frac{2}{\pi}} \frac{\sinh x}{\sqrt{x}}$.

[Note: The function $\Gamma(x)$ is called the gamma function and $I_\nu(x)$ is called the modified Bessel function of the first kind.]

(2) These are second-order ordinary differential equations, where $a > 0$ and n is a positive integer. Find at least one non-trivial solution (i.e., a solution other than $y(x) = 0$) for each equation.

(a) $(a^2 - x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0$ [Hint: Legendre]

(b) $4xy''(x) + 4y'(x) + ay(x) = 0$ [Hint: Bessel; $z = (ax)^{1/2}$].

(c) $y''(x) - 8xy'(x) + 8ny(x) = 0$ [Hint: Hermite]

(3) Find the general solution of **(a)** and **(b)** of the previous problem.

[The exam questions continue on the next page.]

[Part C] Fourier Series Expansion and Fourier Transformation (15 points)

(1) Let n and m are positive integers, $k > 0$, and $L > 0$. Calculate the following integrals with showing the details of the procedure.

(a) $\displaystyle\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$ **(b)** $\displaystyle\int x\sin(kx)\mathrm{d}x$ **(c)** $\displaystyle\int x\cos(kx)\mathrm{d}x$

(2) There are a few conventions for Fourier transform. The textbook defines the Fourier transform of f by $(2\pi)^{-1/2}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}wx}\mathrm{d}x$, but it is also common to define it by $\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi\mathrm{i}wx}\mathrm{d}x$. Here, we use the second convention, i.e., define the Fourier transform by

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi\mathrm{i}wx}\,\mathrm{d}x.$$

(a) Provide the inverse Fourier transform under this convention.

(b) Find the Fourier transform of the function $g(x) = \mathrm{e}^{-x^2}$ under this convention.

[Part D] Extra Problem (unlimited points)

> This represents a sample of small research problems, which you may face in your independent studies. It may not have a definitive solution or may be invalid.

Regular Strum–Liouville problems are summarized on the next page. Sho imagined that the last fact is similar to the Fourier series expansion and we may connect them. So, he made a homework problem, where however he made some mistakes and the idea is still incomplete. Instead of him, try to complete the story based on the following ideas.

- Consider a periodic function $f(x)$ with period 2π; consider only $-\pi \le x \le \pi$.
- Show there is an even function $f_+(x)$ and an odd $f_-(x)$ that satisfy $f(x) = f_+(x) + f_-(x)$.
- Solve the regular Strum–Liouville problem with $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, and

  (A) $k_1 = k_3 = 1$ and $k_2 = k_4 = 0$. **(B)** $k_2 = k_4 = 1$ and $k_1 = k_3 = 0$.

- Then, probably, you may connect the last equation on the next page to the Fourier series expansion.

[This is the end of the exam questions.]