Midterm Exam 112–2 (Apr. 8, 2024)

100 minutes, full mark = 60

Use of your notebooks/memos/books:	You can bring-in only a sheet of A4 paper.
Use of your mobile etc. & Internet:	Strictly forbidden.
Discussion with other attending students:	Strictly forbidden.

Administrative Remarks

- Write your name and student ID on the answer sheet. Put your student ID on the desk.
- Allowed on your desk: student ID card (required), pens/pencils, correction tools (eraser etc.), rulers, drinks, and the brought-in A4 sheet. **Other items must be stored in your bags.**
- **After 13:20, the following acts are regarded as cheating. You may immediately lose your credit.**
  - If non-allowed items (pen cases, foods, poaches, etc.) are found on desks.
  - If your mobile phones, tablets, or PC are not stored in your bags, or if you use them. They must be in your bags even after you submit your answer sheets.
- Breaks are not allowed in principle. After 14:00, you may leave after submission. In case of health problems or other issues, call the TA or lecturer.
- *Any form of academic dishonesty, including chats, additions/corrections after the period, and using your phones, will be treated by NSYSU "Academic Regulations."*

Scientific Remarks

- Include your calculations or thinking process for **partial mark**!
- Use English, where mistakes are tolerated. Meanwhile, scientific mistakes are not tolerated.
- If you notice any errors/issues in the questions, explain the error on your answer sheet, suitably adjust the question, and answer the corrected question.
- You may use the following notations and values without definition/declaration.

$\lvert A\rvert$	determinant of a matrix A (equivalent to $\det A$).
$\lVert\vec{v}\rVert$ or $\lvert\vec{v}\rvert$	norm of a vector $\vec{v}$.
I_n	$n\times n$ identity matrix.
$O_{m,n}$	$m\times n$ zero matrix.
I	identity matrix with shape understood.
O	zero matrix with shape understood.
$\mathbb{R}, \mathbb{C}$	the set of all real/complex numbers.
$\mathbb{R}^n, \mathbb{C}^n$	the set of all n-dimensional real/complex vectors.
$M^{m,n}$	the set of all $m\times n$ real matrices.
$M^{m,n}(\mathbb{C})$	the set of all $m\times n$ complex matrices.
$A \overset{\text{row}}{\sim} B$	A and B are row equivalent.
$A^\dagger = \overline{A^{\mathrm{T}}}$	hermitian conjugate of A.

$\sqrt{2}\approx 1.414$ $\sqrt{3}\approx 1.732$ $\sqrt{5}\approx 2.236$ $\sqrt{7}\approx 2.646$ $\pi\approx 3.142$ $\mathrm{e}\approx 2.718$

Answer **[Part A]**–**[Part D]**. If you still have time, answer **[Part F]**.

[Part A] Matrix Operations (23 points)

Consider the following matrices and vectors, where a, b, w, and θ are real numbers:

$$A=\begin{pmatrix}4&3&2\\0&4&3\\0&0&4\end{pmatrix}\quad D=\begin{pmatrix}1&0&-2&0\\0&1&2&0\\0&1&2&0\end{pmatrix}\quad G=\begin{pmatrix}a&b\\4b&a\end{pmatrix}\quad \vec{a}=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$B=\begin{pmatrix}1&0&1\\0&0&0\\0&0&1\end{pmatrix}\quad E=\begin{pmatrix}1&3&7\\1&-1&-1\end{pmatrix}\quad H=\begin{pmatrix}\cosh w&\sinh w\\\sinh w&\cosh w\end{pmatrix}\quad \vec{b}=\begin{pmatrix}1\\1\\1+\mathrm{i}\end{pmatrix}$$

$$C=\begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix}\quad F=\begin{pmatrix}3&1\\0&2\end{pmatrix}\quad R=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}\quad \vec{c}=\begin{pmatrix}1\\3\\5\end{pmatrix}$$

(1) Which are in row echelon form?

(2) Calculate the following expressions if defined. If it is not defined, answer "not defined".

(a) $A\vec{a}$ **(d)** $\operatorname{rank} A$ **(i)** $\det H$ **(m)** C^{-1}

(b) $((2B^{\mathrm{T}}+B)^{\mathrm{T}}+3B)^{\mathrm{T}}$ **(e)** $\operatorname{rank} D$ **(j)** $\operatorname{rank} H$ **(n)** F^{-1}

(c) $0C\vec{c}$ **(f)** $\operatorname{rank} E$ **(k)** $\det(R^2)$ **(o)** $(\vec{b})^\dagger B\vec{b}$

(3) Calculate $\operatorname{rank} G$.

★**(4)** Consider the linear system of equation, $C\vec{x}=\vec{c}$.

(a) Write down the augmented matrix $\widetilde{C}$.

(b) Find the row echelon form of $\widetilde{C}$ through elementary row operations.

(c) Solve this system.

[Part B] Matrix Diagonalization (10 points)

Consider matrix $A=\begin{pmatrix}\mathrm{i}&1\\-1&\mathrm{i}\end{pmatrix}$ and $B=\begin{pmatrix}-1&2&-2\\2&4&1\\2&1&4\end{pmatrix}$.

Since A is skew-hermitian, it can be diagonalized by a unitary matrix.

★**(1)** Find its eigenvalues and corresponding eigenvectors.

★**(2)** There exists a unitary matrix U such that $U^\dagger AU$ becomes diagonal. Write down U and the diagonal matrix $U^\dagger AU$.

Now, consider B. We know one of its eigenvalues is 5.

(3) Find all the eigenvalues and corresponding eigenvectors of B.

(4) There exists an invertible matrix P such that $P^{-1}BP$ becomes diagonal. Write down P and the diagonal matrix $P^{-1}BP$.

[The exam questions continue on the next page.]

[Part C] Theorems of Linear Algebra (1) (10 points)

Which statements are correct? Choose correct ones from the following, where A, B, C, and D are square matrices. (You do not have to provide any proofs, explanations, or counterexamples.)

(1) $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$.

(2) $(ABCD)^{\dagger} = D^{\dagger}C^{\dagger}B^{\dagger}A^{\dagger}$.

(3) $\det(BA) = \det A \det B$.

(4) $\det(-A) = -\det A$.

(5) Elementary row operations on a square matrix do not change the determinant of the matrix.

(6) Linear system of equations may have infinite number of solutions.

(7) If A and B are row equivalent, $\operatorname{rank} A = \operatorname{rank} B$.

(8) If A is real and symmetric, $-A$ is skew-symmetric.

(9) The eigenvalues of a Hermitian matrix are real.

(10) An $n \times n$ matrix is diagonalizable if and only if its n eigenvalues are all different.

[Part D] Theorems of Linear Algebra (2) (17 points)

(a) The following statements are **correct**. Provide proofs, where you may use the correct (true) statements in the previous problem [Part C] without proving them.

(1) If a matrix A is symmetric as well as skew-symmetric, $A = O$.

★**(2)** Consider an invertible matrix B. Then, $\det(B^{-1}) = (\det B)^{-1}$.

(3) Consider a square matrix C. If $C^2 = C$ and $\det C \neq 0$, then C must be an identity matrix.

★**(4)** If two matrices P and Q are similar, they have the same eigenvalues.

(b) The following statements are **incorrect**. Provide a counterexample for each.

(5) Consider a 2×2 matrix F. If $\det F = 0$ and $\operatorname{Tr} F = 0$, then $F = O$.

(6) Consider 2×2 matrices G and H. If $\operatorname{rank} G = \operatorname{rank} H$, then $\operatorname{rank}(G^2) = \operatorname{rank}(H^2)$.

[Part E] Your Second Chance (a few points)

Problems with ★-marks are the five essential problems in this exam. If you cannot solve some of them during the exam, you can solve them on the blackboard during the discussion hour today (after the exam). You will be helped by other students (and the lecturer) and get some points.

[The exam questions continue on the next page.]

[Part F] Extra Problem (unlimited points)

This is an extra problem, designed for students who had adequate training and complete the other problems very quickly. You can use the following equations:

$$\exp(x) = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{x^k}{k!},$$
$$\exp(\mathrm{i}t) = \cos(t) + \mathrm{i}\sin(t),$$
$$\cos(t) = \lim_{N\to\infty}\mathrm{Re}\sum_{k=0}^{N}\frac{(\mathrm{i}t)^k}{k!} = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{(-1)^k t^{2k}}{(2k)!} = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \cdots,$$
$$\sin(t) = \lim_{N\to\infty}\mathrm{Im}\sum_{k=0}^{N}\frac{(\mathrm{i}t)^k}{k!} = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{(-1)^k t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{6} + \frac{t^5}{120} - \cdots.$$

In physics, we often consider "exponential" of matrices:

$$\exp(\mathrm{i}Ht) := \lim_{N\to\infty}\sum_{k=0}^{N}\frac{(\mathrm{i}Ht)^k}{k!},$$

where H is a square complex matrix and t is a real number. It is known that this limit always converges, so $\exp(\mathrm{i}Ht)$ always exists. Try to calculate

$$\exp\left(\mathrm{i}\sigma_y t\right), \qquad \text{where } \sigma_y = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$$

through the following steps.

(1) Let $A = \mathrm{i}\sigma_y = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. Find the eigenvalues and corresponding eigenvectors of A.

(2) There exists an invertible matrix P such that $D = P^{-1}AP$ becomes diagonal. Write down P and the diagonal matrix n.

(3) Calculate D^n and A^n for non-negative integers $n = 0, 1, 2, 3, \ldots$. For $n = 0$, we define $A^0 = D^0 = I_2$.

(4) Calculate $\dfrac{(At)^{2k}}{(2k)!}$ and $\dfrac{(At)^{2k+1}}{(2k+1)!}$ for non-negative integers k.

(5) Calculate $\exp\left(\mathrm{i}\sigma_y t\right) = \lim_{N\to\infty}\sum_{k=0}^{N}\dfrac{(At)^k}{k!}$.

[This is the end of the exam questions.]