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Received: 17 December 2009 / Received in final form: 29 March 2010 / Accepted: 23 July 2010

Abstract. The two-mode Korteweg-de Vries equation (TMKdV) was proposed to describe the propagation of nonlinear waves of two different wave modes simultaneously. However, the existence of multi-soliton solutions is still unknown. In this letter we present two Hamiltonians, the conservation laws, and a Miura-like transformation of the equation. We show that the TMKdV equation has “quasi-soliton” behaviour in which waves moving in the same direction pass through each other almost without change of their wave forms except for phase shifts.

1 Introduction

The nonlinear two-mode dispersive wave equation was first proposed in [1] to describe the propagation of two different wave modes in the same direction simultaneously. In scaled form it is written as [2]

\[ u_{ttt} - u_{xxx} + (\partial_x - \alpha \partial_x) u u_x + (\partial_x - \beta \partial_x) u_{xxx} = 0, \]  

where \( u(x,t) \) is a field function, \( -\infty < x, t < \infty, -1 \leq \alpha, \beta \leq 1 \), and we refer to it as the two-mode KdV (TMKdV) equation.

Among the related topics come from Hirota and Satsuma [3] who presented a model to describe the interactions of two long waves with different dispersion factors, and found that if there is no effect of one of the waves on the other. As a results, these waves will obey the Korteweg-de Vries (KdV) equation. Gear and Grimshaw [4] studied the interactions between the solitary waves with different wave modes and showed that interaction between the waves occurs when the wave phase speeds are not equal. Moreover, Taninti [5] applied the reductive perturbation method to various wave modes and showed that a nonlinear dispersive wave system can be written as a set of hyperbolic systems of first order equation in time if the higher order derivative terms are neglected.

In the case of TMKdV equation, the existence of multi-soliton solutions is still an open question. It is noted that soliton solution involves four basic ideas: (i) particle-like collisional stability of waves (ii) clean nonlinear interaction between the waves (iii) phase shift after the interaction and (iv) special conserved quantities such as Hamiltonians. The soliton equation being interpreted as a Hamiltonian system is an important aspects of soliton theory [6]. For example, the KdV equation

\[ u_t - 6uu_x + u_{xxx} = 0, \]  

can be written as a Hamiltonian system as

\[ H = \int_{-\infty}^{\infty} \left( u^3 + \frac{1}{2} u_x^2 \right) dx. \]

The Boussinesq equation (BE)

\[ u_t = v_x, \quad v_t = \frac{1}{3} u_{xxx} + \frac{8}{3} u u_x, \]  

can be presented as a Hamiltonian system

\[ \frac{\partial}{\partial t} \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} 0 \\ \partial_x \end{array} \right) \frac{\delta H_0/\delta u}{\delta H_0/\delta v}, \]

where

\[ H_0[u,v] = \int \left( -\frac{1}{6} u_x^2 + \frac{1}{2} u^3 + \frac{1}{2} v^2 \right) dx. \]

In this paper we show three basic conserved quantities, namely the Mass, Momentum and Hamiltonian for the TMKdV equation. We construct two Hamiltonian structures for the equation by using the prolongation method and present a “Miura-like” transformation. We resort to numerical simulations by showing a number of results in order to give a relatively comprehensive understanding of the soliton solutions for the equation. We put two- and three- solitary waves as our initial conditions into our numerical tests, and the numerical results are examined carefully. We show that the TMKdV waves exhibit “quasi-solitons” behaviour- these waves pass through each other almost without change of their wave forms except for phase shifts.
The European Physical Journal Applied Physics

2 The two-mode KdV equation

In this section we want to present its Hamiltonian structure and give full validation for them. We first rewrite it as a system as

\[
\begin{align*}
\dot{u}_t &= q_x, \\
\dot{q}_t &= u_x + \alpha \left( \frac{1}{2} u^2 \right)_x - u q_x + \beta u_{xxx} - q_{xxx},
\end{align*}
\]

(6)

where we have introduced a new variable \( q(x,t) \) and asked that \( \int_{-\infty}^{\infty} u_t \, dx = 0 \) together with the boundary conditions of \( u(x,t), q(x,t) \), their products and derivatives vanishing when \( x \to \pm \infty \). At first sight, (6) simply admits a conservation quantity as

\[
C_1 = \int_{-\infty}^{\infty} u \, dx.
\]

(7)

This can be seen as follows (using the first equation in (6))

\[
\frac{dC_1}{dt} = \int_{-\infty}^{\infty} (q_t + u \dot{u}_t) \, dx = 0,
\]

using the boundary conditions. Its conservation law is \( u_t = q_x \), which is called the conservation of Mass for the TMKdV system.

Similarly, (6) admits another conserved quantity as

\[
C_2 \equiv \int_{-\infty}^{\infty} (q + \frac{1}{2} u^2) \, dx = \text{constant}.
\]

(8)

This is because (using the second equation in (6))

\[
\frac{dC_2}{dt} = \int_{-\infty}^{\infty} (q_t + u \dot{u}_t) \, dx = 0,
\]

using the boundary conditions. Its associated conservation law is represented as

\[
\frac{d}{dt} C_2 + \frac{\partial}{\partial x} P_0 = 0,
\]

(9)

where

\[
P_0 = - \left( u + \frac{\alpha}{2} u^2 + \beta u_{xxx} - q_{xxx} \right).
\]

(10)

Such an expression in (9) is a conservation law for the system, but do not admit the Hamiltonian formulation.

As a result, we have

\[
\frac{\partial}{\partial t} \left( u, q \right) = D \cdot \left( \frac{\delta H}{\delta u}, \frac{\delta H}{\delta q} \right),
\]

(11)

where

\[
D = \begin{pmatrix}
\partial_x, & \partial_x, & -\partial_x, & -\partial_x, \\
\partial_x, & \partial_x & \partial_x, & \partial_x, \\
-\partial_x, & -\partial_x, & -\partial_x, & -\partial_x, \\
-\partial_x, & -\partial_x, & -\partial_x, & -\partial_x,
\end{pmatrix}
\]

(12)

and

\[
H = \int_{-\infty}^{\infty} \left( u q + \frac{1}{2} u^3 - \frac{1}{2} \frac{\partial u^2}{\partial x} \right) \, dx
\]

(13)

is a conserved quantity. Note that although \( u = u(x,t), \)

\( q = q(x,t) \), the functional \( H[u,q] \) is an integral over \( x \), which makes it to depends on \( t \) only.

To further justify the Hamiltonian nature, we first note that a general differential operator \( D \) is of the form

\[
D = \sum_j P_j \partial_j, \quad P_j \in \mathcal{A},
\]

(14)

where \( \mathcal{A} \) be the space of differentiable functions, \( J = 1,2,...,n \), for some finite number \( n \), and \( \partial_j := \partial^j / \partial x^j \). Note that \( \partial_j u \cdot u_x + u \partial_j u \), which is not equal to \( \partial_j u = u_x \). The adjoint \( D^* \) is a differential operator which satisfies

\[
\int P DQ \, dx = \int Q D^* P \, dx, \quad \forall P, Q \in \mathcal{A}.
\]

Corollary: The adjoint of a differential operator can be written as

\[
D^* = \sum_j ( - \partial_j ) \cdot P_j,
\]

(15)

which means that for any \( Q \in \mathcal{A} \), we have

\[
D^* Q = \sum_j ( - \partial_j ) \cdot [P_j Q].
\]

Moreover, an operator \( D \) is self-adjoint if \( D^* = D \), and skew-adjoint if \( D^* = -D \).

When skew-adjoint operator \( D \) is identified, it is possible to define the Poisson structure

\[
\{F,G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u} \frac{\delta G}{\delta \dot{u}} \, dx,
\]

(16)

for any two functionals, where \( \delta / \delta u \) is the variational derivative.

Definition: A differential operator \( D \) is called symplectic (or Hamiltonian) if its Poisson bracket (16) satisfies the “skew-adjoint” property

\[
\{P, Q\} = - \{Q, P\},
\]

(17)

and the “Jacobi identity”

\[
\{P, \{Q, R\}\} + \{\{P, R\}, Q\} = \{\{Q, P\}, R\} = 0,
\]

(18)

for any smooth functionals \( P, Q, \) and \( R \).

The skew-adjoint property of differential operators is easily checked through (15) and Jacobi identity is normally easier to check by examining the closure of the corresponding symplectic form, especially in the case of finite-dimensional systems. However, most of the operators in the infinite-dimensional systems are highly non-trivial, which make it extremely difficult to invert. Thus we will turn to the use of the method of prolongation. We refer the interested readers to [2,7] for details on this method and simply note that if we define a bi-vector as

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\[ \Theta_D = \frac{1}{2} \int \theta \wedge D(\theta) \, dx, \] 
then \( D \) would satisfy the Jacobi identity, i.e., 
\[ \text{pr } \mathbf{v}_D(\Theta_D) = 0, \] 
where \( \text{pr} \) stands for prolongation calculations. Here the assumption is that 
\[ \theta \neq \theta[u], \] 
and by definition, prolongation acts only on coefficients functionally dependent on \( u \).

We now use the theory of functional bi-vector and prolongation of vector field to write its bi-vector form as 
\[
\Theta_D = \int (\theta \wedge \theta_x - \theta \wedge \zeta_{xxx} + 2u \theta_x \wedge \zeta - \zeta \wedge \theta_{xxx} \\
+ \zeta \wedge \zeta_x + \frac{3}{2} \alpha \zeta \wedge \zeta_x + \beta \zeta \wedge \zeta_{xxx} + u^2 \zeta \wedge \zeta_x \\
+ 2u \zeta \wedge \zeta_{xxx} + 2u \zeta \wedge \zeta_{xxx} + 2u \zeta \wedge \zeta_{xxx} \, dx,
\]
where we have used \( \vec{\theta} \equiv (\theta, \zeta) \) as unit vectors for \( u \) and \( q \) respectively, and applied the integration by parts to the vanishing boundary conditions. Next, we need to check its Jacobi identity, meaning that \( \text{pr } \mathbf{v}_D(\Theta_D) = 0 \), where \( \text{pr} \) stands for the prolongation calculation [7]. Hence 
\[
\text{pr } \mathbf{v}_{D \theta}(\Theta_D) = \frac{1}{2} \text{pr } \mathbf{v}_{D \theta} \int (\theta \wedge \theta_x - \theta \wedge \zeta_{xxx} + 2u \theta_x \wedge \zeta \\
- \zeta \wedge \theta_{xxx} + \zeta \wedge \zeta_x + \frac{3}{2} \alpha \zeta \wedge \zeta_x \\
+ \beta \zeta \wedge \zeta_{xxx} + u^2 \zeta \wedge \zeta_x + \zeta \wedge \zeta_{xxx} + 2u \zeta \wedge \zeta_{xxx} + 2u \zeta \wedge \zeta_{xxx} \, dx.
\]

Note further that 
\[
\text{pr } \mathbf{v}_{D \theta} \left( u \theta_x \wedge \zeta \right) \, dx = \int ((\theta_x - \zeta_{xxx} - u \zeta_x \\
- u \zeta_x) \wedge \theta_x \wedge \zeta \, dx,
\]
\[
\text{pr } \mathbf{v}_{D \theta} \left( u \zeta \wedge \zeta_x \right) \, dx = \int ((\theta_x - \zeta_{xxx} - u \zeta_x \\
- u \zeta_x) \wedge \theta_x \wedge \zeta_x \, dx,
\]
\[
\text{pr } \mathbf{v}_{D \theta} \left( u^2 \zeta \wedge \zeta_x \right) \, dx = \int ((2u \theta_x - \zeta_{xxx} - u \zeta_x \\
- u \zeta_x) \wedge \zeta_x \wedge \zeta_x \, dx,
\]
\[
\text{pr } \mathbf{v}_{D \theta} \left( u \zeta \wedge \zeta_{xxx} \right) \, dx = \int ((\theta_x - \zeta_{xxx} - u \zeta_x \\
- u \zeta_x) \wedge \theta_x \wedge \zeta_{xxx} \, dx,
\]
and that 
\[
\int (2 \theta_x \wedge \zeta \wedge \zeta_{xxx}) \, dx = \int (2 \zeta_{xxx} \wedge \theta_x \wedge \zeta \, dx,
\]
\[
\int (-2 u \zeta_x \wedge \theta_x \wedge \zeta) \, dx = - \int (2 u \theta_x \wedge \zeta \wedge \zeta_x \, dx,
\]
\[
\int (-2 u \zeta_{xxx} \wedge \zeta \wedge \zeta_x) \, dx = \int (2 u \zeta_x \wedge \zeta \wedge \zeta_{xxx} \, dx,
\]
we have 
\[
\text{pr } \mathbf{v}_{D \theta}(\Theta_D) = \frac{1}{2} \int \left( \frac{2}{3} \alpha \theta_x \wedge \zeta \wedge \zeta \right) \, dx,
\]
and conclude that system (6) is Hamiltonian if \( \alpha = 0 \). Its conservation law is presented as 
\[
\frac{\partial}{\partial t} \mathcal{H} + \frac{\partial}{\partial x} \mathcal{P} = 0,
\]
where 
\[
\mathcal{P} = -\left( \frac{u^2}{2} + \frac{u^2}{2} + \frac{\alpha u^3}{3} + \beta u u_{xxx} - \frac{\beta u^2}{2} - u q_{xx} \right).
\]
Furthermore, the TMKdV (1) can be cast into another system as 
\[
\begin{cases}
  u_t = (m - \frac{1}{2} u^2 - u x_x)_x, \\
  m_t = (u + \frac{1}{2} \alpha u^3 + \beta u x_x)_x,
\end{cases}
\]
and its Hamiltonian-like form is 
\[
\frac{\partial}{\partial t} \left( \frac{u}{m} \right) = \mathcal{E} \cdot \left( \frac{\delta \mathcal{H}_2 / \delta u}{\delta \mathcal{H}_2 / \delta m} \right),
\]
where 
\[
\mathcal{E} = \left( \frac{\partial u}{\partial x}, 0, \frac{\partial u}{\partial x} + \beta u_x x \right) \cdot \left( \frac{1}{0}, \frac{1}{0} \right),
\]
\[
\mathcal{H}_2 = \int_{-\infty}^{\infty} \left( u m - \frac{1}{6} u^3 + \frac{1}{2} u_x^2 \right) \, dx = \text{constant.}
\]
Here \( \mathcal{E} \) is of the kind same as \( D \) in (12).

Throughout this paper, the operators of the TMKdV equation are defined as \( D, \mathcal{E} : \mathcal{A}^2 \rightarrow \mathcal{A}^2 \), where \( \mathcal{A} \) denotes the space of smooth functions depending on \( x, u \) and derivatives of \( u \) up to some finite order \( n \), denoted for \( (x, u^{(n)}) \in M^{(n)} \). Here \( M^{(n)} \) represent the prolongations of \( M \) in which one first denotes \( M \subset X \times U \) a fixed connected open subset of the space of independent and dependent variables \( x = (x^1, \ldots, x^p) \) and \( u = (u^1, \ldots, u^q) \). \( M^{(n)} \subset X \times U^{(n)} \) are then understood as open subsets of the corresponding jet spaces, with \( (x, u^{(n)}) \in M^{(n)} \) if and only if \( (x, u) \in M \). Hence \( \mathcal{A} \) is denoted by the space of smooth functions \( P(x, u^{(n)}) \) over \( M \) and the functions in \( \mathcal{A} \) are called differential functions (in analogy with the differential polynomials of differential algebra). In particular, \( \mathcal{A} \) is an algebra and its quotient space under image of the total divergence is the space of functionals \( \mathcal{P} = \int P(x, u^{(n)}) \, dx \). Each differential function is a smooth function \( P : M^{(n)} \rightarrow R \) for some finite \( n \). We can write \( P[u] = P(x, u^{(n)}) \) for \( P \), where the square bracket serves to remind us that \( P \) depends on \( x, u \) and derivatives of \( u \). We further define \( \mathcal{A}^l \) to be the vector space of \( l \)-tuples of differential functions, \( P[u] = (P_1[u], \ldots, P_l[u]) \), where each \( P_j \in \mathcal{A} \).

Note that \( \mathcal{A} \) is an algebra, meaning that we can add differential functions and multiply them together. There are also a number of fundamental differential operators on \( \mathcal{A} \) which we have already encountered (e.g., (3) for KdV, \( (5) \) for Boussinesq). Both partial derivatives \( \partial/\partial x^i \) and \( \partial/\partial t \) take a differential function to another differential function, but in general do not preserve the order of derivatives on which they depend. For instances,
\[ P = u_{xxx} + xuu_x \] depends on third order derivatives, but \( \partial P/\partial u = xu_x \) only depends on first order derivatives.

We are able to write the associated bi-vector for \( E \) (23) as

\[
\Theta_E = \frac{1}{2} \int (\theta \wedge \theta_x + \eta \wedge \eta_x + \beta \eta \wedge \eta_{xxx} + \frac{2}{3} \alpha u \eta \wedge \eta_x + \frac{1}{3} \alpha u_x \eta \wedge \eta) \, dx,
\]

where \( \bar{\theta} = (\theta, \eta) \) are unit vectors for \( u, m \) respectively, and by using the prolongation method, we obtain

\[
\begin{align*}
\text{pr} \, v_{E \bar{\theta}}(u) &= \theta_s, \\
\text{pr} \, v_{E \bar{\theta}}(m) &= \eta_x + \beta \eta_{xxx} + \frac{2}{3} \alpha u \eta_x + \frac{1}{3} \alpha u_x \eta. \tag{25}
\end{align*}
\]

Thus we have

\[
\begin{align*}
\text{pr} \, v_{E \bar{\theta}}(\Theta_E) &= \frac{1}{2} \text{pr} \, v_{E \bar{\theta}} \int (\theta \wedge \theta_x + \eta \wedge \eta_x + \beta \eta \wedge \eta_{xxx} + \frac{2}{3} \alpha u \eta \wedge \eta_x + \frac{1}{3} \alpha u_x \eta \wedge \eta) \, dx, \\
\text{pr} \, v_{E \bar{\theta}}(\Theta_E) &= \int \left( \frac{2}{3} \alpha \theta_x \wedge \eta_x \right) \, dx = \int \left( \frac{2}{3} \alpha \theta_x \wedge \eta \wedge \eta_x \right) \, dx, \\
\text{pr} \, v_{E \bar{\theta}}(\Theta_E) &= \int \left( \frac{1}{3} \alpha \eta_{xxx} \wedge \eta \wedge \eta \right) \, dx = \int \left( \frac{1}{3} \alpha \eta_{xxx} \wedge \eta \wedge \eta \right) \, dx = 0.
\end{align*}
\]

Therefore, for general \( \theta, \eta \), we have

\[
\text{pr} \, v_{E \bar{\theta}}(\Theta_E) = \frac{1}{2} \int \left( \frac{2}{3} \alpha \theta_x \wedge \eta_x \right) \, dx = 0. \tag{26}
\]

This shows that (22) is Hamiltonian when \( \alpha = 0 \), and its corresponding conservation law is

\[
\frac{\partial}{\partial \tau} \mathcal{H}_2 + \frac{\partial}{\partial x} P_2 = 0, \tag{27}
\]

where

\[
P_2 = (1/2) \beta u_x^2 - (1/2) u^2 - (\alpha/3) u^3 - \beta uu_{xx} + (1/2) u^2 m - \left( (1/8) u^4 \right) - u_x m_x + m u_{xx} + \left( (1/2) u^2 \right)_x u_x - \left( (1/2) u^2 \right) u_{xx} + u_x u_{xxx} - (1/2) u_x^2.
\]

From our calculation, we note that there is a Miura-like transformation between systems (6) and (22) which is given by

\[
(u, q) \leftrightarrow (u, m) \tag{28}
\]

where

\[
m = q + \frac{1}{2} u^2 + u_{xx}. \tag{29}
\]

This turns the TMKdV equation into the Hamiltonian system and vice versa, which this is the first time it has been reported to exist for such an equation. From our viewpoint, it is very similar to the KdV Miura transformation

\[
u = u^2 + v_x, \tag{30}
\]

which turns the study of the KdV into the study of the modified Korteweg-de Vries equation (MKdV)

\[
u_t - 6u\nu_x + u_{xxx} = (2v + \partial_x) (v_t - 6v^2 v_x + v_{xxx}),
\]

and both equations (KdV, MKdV) all possess Hamiltonian structures.

### 3 Numerics

We apply the Fourier spectral method to numerically integrate equation (6) on a finite interval \( x \in [-L, L] \) with periodic boundary conditions

\[
u(x + 2L, t) = \nu(x, t), \quad q(x + 2L, t) = q(x, t),
\]

and initial conditions

\[
u(x, 0) = \nu_0, \quad q(x, 0) = q_0.
\]

In our numerical scheme, the finite interval \([-128, 128]\) has \( N = 1024 \) grid points with spacing \( \Delta x = 0.25 \), and we use \( \Delta t = 10^{-3} \) to numerically integrate the equation. Our Fourier spectral method is such a pattern: the classical fourth-order Runge-Kutta (RK-4) method in time and spectral derivatives in space [8]. The initial condition is given by a linear sum of two well-separated exact solitary wave solutions of the TMKdV equation as

\[
u(x, 0) = A_1 \text{sech}^2 B_1 (x-x_1) + A_2 \text{sech}^2 B_2 (x-x_2), \tag{31}
\]

\[
q(x, 0) = -\lambda_1 A_1 \text{sech}^2 B_1 (x-x_1) - \lambda_2 A_2 \text{sech}^2 B_2 (x-x_2),
\]

where

\[
A_i = 3 \left( \lambda_i^2 - 1 \right) / (\lambda_i + \alpha), \tag{32}
\]

\[
B_i = (1/2) \sqrt{\left( \lambda_i^2 - 1 \right) / (\lambda_i + \beta)}, \tag{33}
\]

for \( i = 1, 2 \). The solution profiles for the TMKdV equation are presented in Figures 1–3. We note that by considering \( \alpha = 0, \beta = 0.2 \) for the TMKdV equation, we are dealing with both the Hamiltonian system and its soliton phenomenon at the same time. One can see that TMKdV taller solitary wave catches the shorter, coalesces to form a single wave, then reappears in front of the shorter wave (to the right). The interaction is nonlinear, not simply the superposition of two individual waves and but still appears soliton-like phenomenon. We also compute the Hamiltonian in (13) to give full validation of the accuracy of our numerical scheme. The Hamiltonian check in Figure 4 is as accurate as of \( O \left( 10^{-13} \right) \), showing that our results are genuine.

Solutions are further depicted and enlarged at four different times and shown in Figure 5 where some dispersive wave components were observed moving in the direction away from the main wave solutions. We go on to test the re-emergence of the solitary waves after collisions. What we are interested in is the accuracy of the numerical values of \( u(x, t) \) produced by the numerical scheme. It was found that the taller solitary wave amplitude is dropped from its maximum value (height) of 3.7339 to 3.7333, and the smaller solitary wave amplitude is dropped from its maximum value (height) of 1.1000 to 1.0918, making the relative errors being as \( \approx 0.016\% \) and \( \approx 0.745\% \) respectively.
Fig. 1. (Color online) (a) TMKdV solution profiles solutions, the initial condition is set up with \( \lambda_1 = 1.8 \), \( \lambda_2 = 1.2 \), \( x_1 = -75 \) and \( x_2 = -50 \) for waves in (31).

Fig. 2. Contour plot of interacting solitary waves.

Fig. 3. Comparison with linear waves at the final time of calculation \( t = 80 \), showing phase shifts.

Fig. 4. Numerical Hamiltonian in (13), the values are presented as differences from the initial value.

4 Three-soliton like interaction

By analogy with the two solitary waves case, we perform the three solitary waves interaction on a collision course. Care is taken to choose the initial positions of the waves in an appropriate way, otherwise we would be merely creating a series of two-soliton collisions in the presence of a nearby third solitary wave. We use the same numerical settings as above and initial conditions are given by a linear sum of three well-separated solitary waves with different amplitudes and velocities,

\[
\begin{align*}
    u(x, 0) &= \sum_{i=1}^{3} A_i \text{sech}^2 B_i (x - x_i), \\
    q(x, 0) &= \sum_{i=1}^{3} (-\lambda_i) A_i \text{sech}^2 B_i (x - x_i),
\end{align*}
\]

where \( A_i, B_i \), \( i = 1, 2, 3 \) are assumed as in (32), (33). We consider \( \alpha = 0, \beta = 0.2, \lambda_1 = 1.8, \lambda_2 = 1.45, \) and \( \lambda_3 = 1.08 \) for the initial conditions (34), (35) so that \( A_i \) and \( B_i \) are all real numbers and we use \( x_1 = -34, x_2 = -14, x_3 = 7 \) to separate these waves so that the faster waves has the chance to overtake the slower waves in our numerical simulation. The numerical results are shown in Figures 6–10, indicating that the “collisional stability” observed in the two-soliton-like collision applies equally well in the three-soliton-like case. However, further check on the waves show that the largest wave amplitude has dropped from a maximum of 3.7333 to 3.7325, the second largest wave amplitude is dropped from a maximum of 2.2819 to 2.2673, and the smallest wave amplitude is dropped from a maximum of 0.4690 to 0.4622, with relative errors being as \( \approx 0.021\% \), \( \approx 0.639\% \) and \( \approx 1.445\% \).
respectively. In order to see if the change in the heights of the solitons is numerical artificial, we conduct a series of numerical programs by keeping halving the space discretizations $\Delta x$ and temporal discretizations $\Delta t$ and compare the obtained results. Such an idea comes from Lax and Richtmyer [9] in 1956 when they observed the stability problem of the numerical solutions is actually the problem that characterizes the convergence of the discrete approximations to the correct solution when the mesh is refined. We run the numerical programs for two-soliton solutions up to time $t = 80$, and compute the relative errors from two successive numerical experiments. The first numerical results are shown in Figure 11 in which the maximum errors of solutions are seen to keep about $O(10^{-13})$, showing that the initial time step that we chooses has already rendered the temporal errors negligible. The second
Careful investigation of numerical solutions for refined meshes $\Delta x = 0.125$ and $\Delta x = 0.0625$ show no change in the heights of waves compared with those obtained for $\Delta x = 0.25$. This shows that the TMKdV equation does not have pure solitons but “quasi-solitons”. Note that in Figure 8, we see the wave positions have been changed from their “linear” locations, resulting in the phase shifts. This is due to the nonlinear interaction between the waves.

Our results shows that these TMKdV wave solutions are real, not numerically artificial, and small disturbances and dispersive-wave component are physical and visible which indicates that the TMKdV equation experiences an imperfect elastic collisions compared to the usual kind of the KdV soliton behaviour and hence can be regarded as the “quasi-soliton” solution.
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